

# WHY IS CPT FUNDAMENTAL?

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## Abstract

G. Lüders and W. Pauli proved the  $CPT$  theorem based on Lagrangian quantum field theory almost half a century ago. R. Jost gave a more general proof based on “axiomatic” field theory nearly as long ago. The axiomatic point of view has two advantages over the Lagrangian one. First, the axiomatic point of view makes clear why  $CPT$  is fundamental—because it is intimately related to Lorentz invariance. Secondly, the axiomatic proof gives a simple way to calculate the  $CPT$  transform of any relativistic field without calculating  $\mathcal{C}$ ,  $\mathcal{P}$  and  $\mathcal{T}$  separately and then multiplying them. The purpose of this pedagogical paper is to “deaxiomatize” the  $CPT$  theorem by explaining it in a few simple steps. We use theorems of distribution theory and of several complex variables without proof to make the exposition elementary.

## 1 Introduction

The notion of  $CPT$  symmetry, where  $\mathcal{C}$  is charge conjugation,  $\mathcal{P}$  is parity (space inversion) and  $\mathcal{T}$  is time reversal in the sense of Wigner,<sup>2</sup> as a symmetry that holds for any relativistic quantum field theory evolved from the observation of G. Lüders [1] that charge conjugation symmetry and spacetime inversion symmetry both impose the same constraints on the form of the interaction Hamiltonian so that  $CPT$  symmetry has a more fundamental basis than either  $\mathcal{C}$ ,  $\mathcal{P}$  or  $\mathcal{T}$ . W.

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<sup>2</sup> $\mathcal{C}$  and  $\mathcal{P}$  are unitary;  $\mathcal{T}$  and  $CPT$  are antiunitary

Pauli [2] gave a clear formulation of  $\mathcal{CPT}$  symmetry in the context of conditions on the interaction Hamiltonian or Lagrangian. Pauli's formulation is the form of the  $\mathcal{CPT}$  symmetry that is usually discussed, the “Lagrangian  $\mathcal{CPT}$  theorem.” R. Jost [3] gave a general proof of  $\mathcal{CPT}$  symmetry based on the fact that spacetime inversion is connected to the identity in the complex Lorentz group although this inversion is not connected to the identity in the real Lorentz group. Jost's analysis is usually called the “axiomatic  $\mathcal{CPT}$  theorem.” J. Schwinger [4] discussed the  $\mathcal{CPT}$  and spin-statistics theorems from the point of view of his differential action principle.

Jost's proof has been labeled as belonging to axiomatic field theory as though that made his proof both incomprehensible and of no practical value. In fact Jost's proof is easy to understand if one is willing to accept theorems about distributions and analytic functions of several complex variables without proof. To make notation simple in this pedagogical paper I have exercised all the test functions that usually appear in discussions of singular functions (distributions). To make clear my lack of rigor I have used the word “analytic” rather than the word “holomorphic” in connection with the functions of several complex variables that appear.

Jost's proof has the practical value that it gives a very simple and general result for the  $\mathcal{CPT}$  symmetry acting on any relativistic quantum field. Jost discussed the  $\mathcal{CPT}$  theorem in three publications, his original paper in Helvetica Physica Acta [3], his contribution (in German) to the Pauli memorial volume [5] and his book [6] on quantum field theory. Jost's theorem also is discussed in the books by R. Streater and A.S. Wightman [7], N.N. Bogoliubov, A.A. Logunov and I.T. Todorov [8], and R. Haag [9].

The standard textbooks of quantum field theory all get to the  $\mathcal{CPT}$  theorem by calculating each symmetry and then calculating their product. This is not incorrect (except for the technical fact that each of  $\mathcal{C}$ ,  $\mathcal{P}$  and  $\mathcal{T}$  can have an arbitrary phase since they are not connected to the identity while  $\mathcal{CPT}$ , which is connected to the identity, cannot have an arbitrary phase). However calculating  $\mathcal{CPT}$  by multiplying each of the three discrete symmetries is a very complicated way to calculate  $\mathcal{CPT}$ . More important, calculating  $\mathcal{CPT}$  in that way obscures why  $\mathcal{CPT}$  is fundamental but none of the individual symmetries is.

The purpose of this expository note is to explain why  $\mathcal{CPT}$  is fundamental

and to calculate it for a general relativistic quantum field without worrying about the mathematical issues connected with functions of several complex variables and their relation to tempered distributions whose support in momentum space lies in or on a cone.  $\mathcal{CPT}$  is fundamental because it is closely related to Lorentz covariance. We will pay attention to how far we can get with Lorentz covariance alone and where we must use another property of the theory. The reader will also see that the calculation of  $\mathcal{CPT}$  using general arguments is greatly simpler than the pedestrian calculation of  $\mathcal{C}$ ,  $\mathcal{P}$  and  $\mathcal{T}$  separately and then multiplying them. To make this note self contained we will explain ideas connected with group theory and field theory that many readers will already understand. Those readers are encouraged to skip the introductory explanations and go directly to the  $\mathcal{CPT}$  theorem itself.

## 2 Representations of the real and complex Lorentz groups

Since the heart of the argument is the fact that the connected component of the complex Lorentz group,  $L(C)$ , which is the proper complex Lorentz group,  $L_+(C)$ , contains spacetime inversion, we will discuss the Lorentz group first [6]. The (real) Lorentz group can be taken as the group,  $SO(1,3)$ , of real 4 x 4 matrices  $\Lambda$  that preserve the metric  $g$  that we take to have the form  $g = \text{diag}(1, -1, -1, -1)$ ,

$$\Lambda^T g \Lambda = g. \quad (1)$$

You can check that this condition is equivalent to saying that a Lorentz transformation preserves the scalar product  $x^2 = x \cdot x = (x^0)^2 - \sum_1^3 (x^i)^2$ , i.e.,  $\Lambda x \cdot \Lambda x = x \cdot x$ . By taking the determinant of Eq.(1) we see that  $\det \Lambda = \pm 1$ . By looking at the 00 element of Eq.(1) we find  $(\Lambda^0_0)^2 - \sum_1^3 (\Lambda^0_i)^2 = 1$ , so either  $\Lambda^0_0 \geq 1$  or  $\Lambda^0_0 \leq -1$ . Thus the Lorentz group falls into four disconnected components,  $L_+^\uparrow$ ,  $L_+^\downarrow$ ,  $L_-^\uparrow$ , and  $L_-^\downarrow$  according to the sign of the determinant of  $\Lambda$  and the sign of  $\Lambda^0_0$ . Only the first of these is a group since only  $L_+^\uparrow$  contains the identity. We use  $x \in V_+$  if  $x^2 > 0, x^0 > 0$ ;  $x \in V_-$  if  $x^2 > 0, x^0 < 0$ ,  $x \sim 0$  if  $x^2 < 0$ . We also have to consider the complex Lorentz group, the group of complex 4 x 4 matrices that obey Eq.(1). For the complex Lorentz group the sign of the determinant still cannot be changed

continuously, but the matrix  $-1$  is now connected to the identity, so there are only two disconnected components. The easiest way to find the continuous family of complex Lorentz transformations that connect the matrices  $1$  and  $-1$  is by considering the covering groups of the real and complex Lorentz groups, to which we now turn.

We are familiar with the fact that a spin  $1/2$  state transforms under a rotation by an angle  $\theta$  with a phase  $\theta/2$  rather than the phase  $\theta$  of a scalar state. So a rotation by  $2\pi$  changes the phase of a spin  $1/2$  state even though such a rotation should be equivalent to the identity. Thus a spin  $1/2$  state does not transform as a true representation of the rotation group, but rather as a representation up to a factor. The idea of a covering group is to find a larger group whose representations are true representations without additional phases. For the rotation group the covering group is  $SU(2)$ , the group of  $2 \times 2$  unitary complex matrices with determinant 1. For the connected component of the Lorentz group the covering group is  $\bar{L}_+^\uparrow \equiv SL(2, C)$ , the group of  $2 \times 2$  complex matrices of determinant 1.

We introduce the two fundamental representations of  $SL(2, C)$  as

$$u'_\alpha = A_{\alpha\beta} u_\beta \quad (2)$$

and

$$\dot{v}'_{\dot{\alpha}} = A_{\dot{\alpha}\dot{\beta}}^* \dot{v}_{\dot{\beta}}, \quad (3)$$

where  $A \in SL(2, C)$  and  $*$  stands for complex conjugate. Spinors with undotted and dotted indices were introduced in [10]. We can introduce a scalar product for these representations using the  $2 \times 2$  antisymmetric Levi-Civita symbol  $\epsilon_{\alpha\beta}$  for the undotted spinors and  $\epsilon_{\dot{\alpha}\dot{\beta}}$  for the dotted spinors. We choose  $\epsilon_{12} = 1$ ,  $\epsilon_{\dot{1}\dot{2}} = 1$ . Any representation of  $SL(2, C)$  has the form of a spinor with  $k$  undotted and  $l$  dotted indices, each transforming as given above. The only way we can reduce these representations is by contracting with the  $\epsilon$ 's just described, so the irreducible representations of  $SL(2, C)$  are spinors with  $k$  symmetrized undotted and  $l$  symmetrized dotted indices. Each index corresponds to spin  $1/2$  so these spinors have spin  $k/2$  and  $l/2$  under the  $SU(2) \otimes SU(2)$  formed by taking the groups whose generators are  $J \pm iK$ , where  $J$  are the rotation generators of the real Lorentz group and  $K$  are the generators of pure Lorentz transformations (boosts). (See the appendix for this description.)

We take the Pauli matrices to have one undotted and one dotted index,  $(\sigma_\mu)_{\alpha}^{\dot{\beta}}$ , where  $\sigma_0$  is the unit  $2 \times 2$  matrix and  $\sigma_i$  are the usual Pauli matrices. Then we can uniquely associate a  $2 \times 2$  hermitian matrix  $X$  with a real vector  $x^\mu$  by  $(X)_{\alpha}^{\dot{\beta}} = x^\mu (\sigma_\mu)_{\alpha}^{\dot{\beta}}$ . To invert this, trace with the  $\sigma$  matrices. The reader should check that  $\det X = x^2$ . Recalling that matrices in  $SL(2, C)$  have determinant 1, we see that  $X' = AXA^\dagger$  is again Hermitian and is a Lorentz transformation on  $x$ , where  $\dagger$  stands for adjoint. The matrices  $A$  and  $-A$  stand for the same Lorentz transformation; thus the group  $SL(2, C)$  covers the connected component of the Lorentz group twice.

To cover the complex Lorentz group we allow two independent  $SL(2, C)$  matrices to enter so that  $X' = AXB^T$ . This  $X'$  is no longer Hermitian, but it still has the same Minkowski metric length, so the covering group of the complex Lorentz group,  $L_+(C)$ , is  $SL(2, C) \otimes SL(2, C)$ . Since now we have two independent matrices  $A$  and  $B$  at our disposal, we can achieve  $x \rightarrow -x$  either by choosing  $A = 1$ ,  $B = -1$  or  $A = -1$ ,  $B = 1$ . We can go continuously from the identity  $A = 1$ ,  $B = 1$  to  $A = 1$ ,  $B = -1$  in the first case by choosing  $A = 1$ ,  $B(\phi) = \text{diagonal}(\exp i\phi/2, \exp -i\phi/2)$ . We can find the  $4 \times 4$  complex Lorentz transformations  $\Lambda(\phi)$  from the definition of  $X'$ . The result, which is a continuous family of complex Lorentz transformations going from the identity to spacetime inversion, is

$$\begin{pmatrix} x'^0 \\ x'^3 \\ x'^1 \\ x'^2 \end{pmatrix} \begin{pmatrix} \cos \frac{\phi}{2} & i \sin \frac{\phi}{2} & 0 & 0 \\ i \sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cos \frac{\phi}{2} & -\sin \frac{\phi}{2} \\ 0 & 0 & \sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix} \begin{pmatrix} x^0 \\ x^3 \\ x^1 \\ x^2 \end{pmatrix}$$

Are there other ways to achieve spacetime inversion? In  $SL(2, C) \otimes SL(2, C)$  we need  $AXB^T = -X$ , or  $AX = -XB^{T-1}$ . Thus we need this relation where  $X$  is replaced by each of the Pauli matrices  $\sigma_\mu$ . For  $\sigma_0 = 1$  we need  $A = -B^{T-1}$ . Then we need  $AX = XA$  where for  $X$  we can choose any of the space  $\sigma$ 's. This requires  $A = \omega 1$  and for  $A \in SL(2, C)$  we need  $\omega^2 = 1$  or  $\omega = \pm 1$ . Thus the *only* possibilities to invert  $x^\mu$  are the ones given above. Now we have the group theory we need to discuss the  $\mathcal{CPT}$  theorem.

### 3 Vacuum matrix elements of products of fields define analytic functions.

Next we have to discuss vacuum matrix elements of products of fields, often called Wightman functions or distributions. Let  $\phi^{(k,l)}(x)$  be a field with  $k$  undotted and  $l$  dotted indices, each set symmetrized, that transforms as the irreducible representation of  $SL(2, C)$  described above. We will use the active point of view in which a Poincaré transformation  $(a, A)$  acts as<sup>3</sup>

$$U(a, A)\phi^{(k,l)}(x)U(a, A)^\dagger = S^{(k,l)}(\Lambda)^{(-1)}\phi^{(k,l)}(\Lambda x + a); \quad (4)$$

The only case for which we need the detailed form of  $S^{(k,l)}(A, B)$  is when  $\Lambda \in L_+(C)$  produces spacetime inversion and for that case  $S^{(k,l)}(A, B)$  is just a diagonal phase. Thus the detailed form of  $S^{(k,l)}$  is not necessary here. For this reason we have suppressed the indices belonging to the matrices  $S^{(k,l)}$  as well as the indices belonging to the field  $\phi^{(k,l)}(x)$ .<sup>4</sup> We assume the vacuum  $|0\rangle$  is invariant under Poincaré transformations,

$$U(a, A)|0\rangle = |0\rangle. \quad (5)$$

We were tempted first to use scalar fields in discussing Jost's proof in order to avoid cumbersome notation and then to give the argument again for the general case. Instead, in order to make clear how simple Jost's argument is, we decided to streamline the notation instead and give the general case directly. (For some properties such as the support in momentum space which does not depend on the spin we will use the scalar case to illustrate the issue.) Let the single index  $(p)$  (for "pair" of indices) stand for  $(k, l)$ . We will use  $(p)$  and  $(k, l)$  interchangeably to label fields and other objects. Then the general field becomes  $\phi^{(p)}(x)$ , the matrices are

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<sup>3</sup>We are using the covering group of the Poincaré group, so in  $(a, \Lambda)$  we replaced  $\Lambda \in L_+^\uparrow$  by  $A \in SL(2, C)$ . In the argument of the fields on the right hand side on the next equation we replaced  $\Lambda$  by  $\Lambda(A)$  where  $\Lambda(A) \in L_+^\uparrow$  is the homomorphic image of  $A \in SL(2, C)$ . Where we use the the covering group of the complex Lorentz group we should replace  $\Lambda \in L_+(C)$  by  $\Lambda(A, B)$  where  $\Lambda(A, B)$  is the homomorphic image of  $(A, B) \in SL(2, C) \otimes SL(2, C)$ . To simplify notation we will write  $\Lambda$  instead of  $\Lambda(A)$  or  $\Lambda(A, B)$  in both cases.

<sup>4</sup>See the appendix for the transformations with all indices exhibited.

$S^{(p)}(A)$ , and the transformation law, again suppressing indices, is

$$U(a, A)\phi^{(p)}(x)U^\dagger(a, A) = S^{(p)}(A)^{-1}\phi^{(p)}(\Lambda x + a). \quad (6)$$

Next we write the vacuum matrix element of an arbitrary product of fields and use this transformation law to find

$$\begin{aligned} & \langle 0 | \phi^{(p_1)}(x_1) \phi^{(p_2)}(x_2) \cdots \phi^{(p_n)}(x_n) | 0 \rangle \\ &= \langle |0\rangle, \phi^{(p_1)}(x_1) \phi^{(p_2)}(x_2) \cdots \phi^{(p_n)}(x_n) | 0 \rangle \rangle \\ &= \langle U(a, A) | 0 \rangle, U(a, A) \phi^{(p_1)}(x_1) \phi^{(p_2)}(x_2) \cdots \phi^{(p_n)}(x_n) | 0 \rangle \rangle \\ &= \langle |0\rangle, [\prod_1^n S^{(p_i)}(A)^{-1}] \phi^{(p_1)}(\Lambda x_1 + a) \phi^{(p_2)}(\Lambda x_2 + a) \cdots \phi^{(p_n)}(\Lambda x_n + a) | 0 \rangle \rangle \\ &= [\prod_1^n S^{(p_i)}(A)^{-1}] \langle 0 | \phi^{(p_1)}(\Lambda x_1 + a) \phi^{(p_2)}(\Lambda x_2 + a) \cdots \phi^{(p_n)}(\Lambda x_n + a) | 0 \rangle. \end{aligned} \quad (7)$$

Because of translation invariance this matrix element depends on only  $n - 1$  differences of the spacetime coordinates. We define the Wightman function [11] which is a generalized function or distribution

$$W^{(n; p_1 p_2 \cdots p_n)}(x_1 - x_2, x_2 - x_3, \cdots, x_{n-1} - x_n) \equiv \langle 0 | \phi^{(p_1)}(x_1) \phi^{(p_2)}(x_2) \cdots \phi^{(p_n)}(x_n) | 0 \rangle. \quad (8)$$

Since we will have to deal with three kinds of difference vectors, we will use different letters to distinguish them:  $\xi$  for real vectors,  $\rho$  for real vectors, called “Jost points” defined below, in the domain of analyticity  $\mathcal{T}'_{n-1}$ , and  $\zeta$  for complex vectors. To streamline the notation we compress the indices  $(p_1 p_2 \cdots p_n)$  on  $F^{(n)}$  to a single index  $(\wp)$ . We define  $\xi_j = x_j - x_{j+1}$ . Then invariance under the connected component of the Lorentz group (the proper orthochronous component,  $L_+^\uparrow$ ) gives

$$W^{(n; \wp)}(\Lambda \xi_1, \Lambda \xi_2, \cdots, \Lambda \xi_{n-1}) = [\prod_1^n S^{(p_i)}(A)] W^{(n; \wp)}(\xi_1, \xi_2, \cdots, \xi_{n-1}). \quad (9)$$

The requirement that physical states have positive energy, except the vacuum which has zero energy, implies that the momenta in the Fourier transform of the  $F^{(n; \wp)}$ 's lie in or on the forward light cone. To see this we drop all indices and consider a

matrix element of a scalar field, since the support in momentum space depends only on the translation subgroup of the Poincaré group,

$$\langle 0 | \phi(x_1) \cdots \phi(x_j) U(a, 1) \phi(x_{j+1}) \cdots \phi(x_n) | 0 \rangle. \quad (10)$$

Evaluate this either by letting the translation operator act to the right to get

$$W^{(n)}(\xi_1, \dots, \xi_{j-1}, \xi_j + a, \xi_{j+1}, \dots, \xi_{n-1}), \quad (11)$$

or by inserting the identity operator using a complete set of intermediate states,  $|q_j, \alpha_j\rangle$ , before the translation operator to get

$$\langle 0 | \phi(x_1) \cdots \phi(x_j) \sum_{q_j, \alpha_j} |q_j, \alpha_j\rangle \langle q_j, \alpha_j | \exp(-iq_j \cdot a) \phi(x_{j+1}) \cdots \phi(x_n) | 0 \rangle. \quad (12)$$

This shows that the physical momenta  $q_j$  are conjugate to the spacetime difference vectors  $\xi_j$ . Thus

$$W^{(n)}(\xi_1, \xi_2, \dots, \xi_{n-1}) = \frac{1}{(2\pi)^{4(n-1)}} \prod_1^{n-1} \int d^4 q_j \exp(-iq_j \cdot \xi_j) \tilde{W}^n(q_1, \dots, q_{n-1}), \quad (13)$$

where the momenta  $q_j$  are physical, i.e., are in or on the forward light cone.

Now use the intuitive criterion that a Fourier transform that is a distribution becomes an analytic function when the external variable is made complex in such a way as to provide a damping factor so that the Fourier transform becomes a Laplace transform. Thus we must examine when the factor  $\exp(-iq_j \cdot \xi_j)$  becomes decreasing for  $\xi_j \rightarrow \zeta_j = \xi_j + i\eta_j$  with  $\xi_j$  and  $\eta_j$  real vectors. What counts is the absolute value of the factor which is  $\exp(q_j \cdot \eta_j)$ . This becomes decreasing if  $\eta_j$  is in or on the backward light cone since the physical momentum  $q_j$  is in the forward light cone. Thus the Wightman function

$$W^{(n)}(\zeta_1, \zeta_2, \dots, \zeta_{n-1}) \quad (14)$$

is an analytic function of  $4(n-1)$  complex variables (in four-dimensional spacetime) when  $Im\zeta_j \in V_-^\uparrow$ . It is also single-valued.



## 4 Enlargement of the domain of analyticity

As it stands,  $W^{(n)}$  is analytic only when  $Im\zeta_i \neq 0$ , i.e., its domain of analyticity has no real points. Call this domain, which has the form of a tube with  $Re\zeta_i$  arbitrary and  $Im\zeta_i ina \in V_-$ , the tube  $\mathcal{T}_{n-1}$ . Now we restore the labels of the fields and use a profound result due to V. Bargmann, D.W. Hall and A.S. Wightman [12] which applies to this case. If

$$W^{(n;\varphi)}(\Lambda\zeta_1, \Lambda\zeta_2, \dots, \Lambda\zeta_{n-1}) = [\prod_1^n S^{(p_i)}(A)]W^{(n;\varphi)}(\zeta_1, \zeta_2, \dots, \zeta_{n-1}) \quad (15)$$

for the covering group,  $SL(2, C)$ , of real  $\Lambda \in L_+^\uparrow$  then  $W^{(n;\varphi)}(\zeta_1, \zeta_2, \dots, \zeta_{n-1})$  has a (unique) single-valued analytic continuation to the domain  $\mathcal{T}'_{n-1}$ , that we call the extended tube,<sup>5</sup> that is the union of all  $\Lambda\mathcal{T}_{n-1}$  where now we have complex matrices  $\Lambda \in L_+(C)$ . This enlargement of the domain of analyticity leads to two crucial results. First, in contrast to  $\mathcal{T}_{n-1}$ , our new, larger domain of analyticity,  $\mathcal{T}'_{n-1}$ , contains real points of analyticity ( $\rho_j$ ) that we will discuss below. Secondly since  $\mathcal{T}'_{n-1}$  is invariant under complex Lorentz transformations in  $L_+(C)$ , one of which is spacetime inversion, we have the relation

$$W^{(n;\varphi)}(\zeta_1, \zeta_2, \dots, \zeta_{n-1}) = (-1)^L W^{(n;\varphi)}(-\zeta_1, -\zeta_2, \dots, -\zeta_{n-1}) \quad (16)$$

$L = \sum l_i$ , in  $\mathcal{T}'_{n-1}$ . To see where the factor  $(-1)^L$  comes from we repeat that for  $\Lambda \in L_+^\uparrow$ ,  $\Lambda$  depends on the matrices  $A$  and  $A^*$  in  $SL(2, C)$ ; however, as mentioned in a footnote above, now that we have the extension to  $L_+(C)$ ,  $\Lambda$  depends on two independent matrices  $A$  and  $B$  in  $SL(2, C)$  and we can transform continuously from the identity to spacetime inversion. The  $S^{(k,l)}$  matrices for spacetime inversion are just powers of  $(-1)$ ; thus if we choose  $A = 1$ ,  $B = -1$  then  $S^{(k,l)}(1, -1) = (-1)^l \mathbf{1}$ <sup>6</sup> Note that spacetime inversion here is an element of the complex Lorentz group and is unitary, not antiunitary.

Jost found a precise characterization of the real points,  $(\rho_j)$ , in  $\mathcal{T}'_{n-1}$ :  $\sum_1^{n-1} \lambda_i \rho_i \sim 0$ , for all  $\lambda_i > 0$  such that  $\sum_1^{n-1} \lambda_i \geq 0$ . This requires that each  $\rho_i \sim 0$ .

<sup>5</sup>although technically it is not a tube in the variables  $(\zeta_1, \dots, \zeta_{n-1})$

<sup>6</sup>Here  $\mathbf{1}$  is the direct product of a  $(2k+1) \times (2k+1)$  unit matrix and a  $(2l+1) \times (2l+1)$  unit matrix and we find the result just stated for  $L$ .

Jost's result is particularly simple for the  $W^{(2)}$  function for single scalar field in which there is one complex difference vector  $\zeta$ . Since  $W^{(2)}(\Lambda\zeta) = W^{(2)}(\zeta)$  we can find the extended tube  $\mathcal{T}'_1$  by finding the values of  $\zeta^2$  that can be obtained from  $\Lambda\zeta$  with  $\zeta = \xi + i\eta$ ,  $\eta \in V_-$ . Then  $\zeta^2 = \xi^2 - \eta^2 + 2i\xi \cdot \eta$ . The real points are those for which  $\xi \cdot \eta = 0$ , with  $\eta \in V_-$ . These points are the spacelike points  $\xi \sim 0$ , in agreement with Jost's general result.

## 5 The general formula for $\mathcal{CPT}$ .

When we write the relation between Wightman functions at Jost points that comes from spacetime inversion, Eq.(4), in terms of vacuum matrix elements we find

$$\begin{aligned} \langle 0 | \phi^{(k_1, l_1)}(x_1) \phi^{(k_2, l_2)}(x_2) \cdots \phi^{(k_n, l_n)}(x_n) | 0 \rangle = \\ (-1)^L \langle 0 | \phi^{(k_1, l_1)}(-x_1) \phi^{(k_2, l_2)}(-x_2) \cdots \phi^{(k_n, l_n)}(-x_n) | 0 \rangle. \end{aligned} \quad (17)$$

Each side of this last equation can be analytically continued; the left hand side to complex  $\zeta_i$  with  $\eta = Im\zeta \in V_+^\perp$  and the right hand side to complex  $\zeta_i$  with  $\eta = Im\zeta \in V_+^\perp$ . We cannot take the limit as  $\eta_i \rightarrow 0$  to get a relation between vacuum matrix elements of products of the fields, because, as just noted, if  $\zeta_i$  has its imaginary part in the backward cone, then  $-\zeta_i$  has its imaginary part in the forward cone and then the analytic continuations of the functions on the two sides of Eq.(4) are not valid in the same domain. On the other hand if we consider the vacuum matrix element with the fields in completely reversed order,

$$\langle 0 | \phi^{(p_n)}(-x_n) \cdots \phi^{(p_2)}(-x_2) \phi^{(p_1)}(-x_1) | 0 \rangle, \quad (18)$$

which corresponds to

$$W^{(n; i\varphi)}(\xi_{n-1}, \cdots, \xi_2, \xi_1), \quad (19)$$

in terms of difference variables, where  $i\varphi$  stands for  $(p_n \cdots p_2 p_1)$ , both functions will have the same domain,  $\mathcal{T}'_{n-1}$ , of analyticity. This is precisely where we have to assume something beyond Lorentz covariance. To reverse the order of all the fields we assume that at a Jost point the two vacuum matrix elements are related by a sign  $(-1)^I$  where  $I$  is the number of transpositions of Fermi fields necessary to

invert the order of the fields. This is implied by the spin-locality theorem<sup>7</sup> but is weaker than that theorem since we need this relation only at Jost points (or even only in a neighborhood of a Jost point) in each matrix element, rather than as an operator relation. If there are  $F$  Fermi fields then  $I = (F - 1) + (F - 2) + \cdots + 1 = 1/2F(F - 1)$ . Since the number of Fermi fields in a nonvanishing vacuum matrix element must be even,  $F - 1$  must be even, thus the phase that enters is  $(-1)^{1/2F(F-1)} = ((-1)^{(F-1)})^{F/2} = (-1)^{F/2} = i^F$ . The condition on matrix elements that

$$\begin{aligned} \langle 0 | \phi^{(k_1, l_1)}(x_1) \phi^{(k_2, l_2)}(x_2) \cdots \phi^{(k_n, l_n)}(x_n) | 0 \rangle = \\ i^F \langle 0 | \phi^{(k_1, l_1)}(x_n) \cdots \phi^{(k_2, l_2)}(x_2) \phi^{(k_n, l_n)}(x_1) | 0 \rangle \end{aligned} \quad (20)$$

at Jost points is called “weak local commutativity.” Clearly local commutativity (sometimes called microcausality) implies weak local commutativity. Combining spacetime inversion and weak local commutativity and collecting phases, we have

$$W^{(n; \varphi)}(\zeta_1, \zeta_2, \cdots, \zeta_{n-1}) = i^F (-1)^L W^{(n; i\varphi)}(\zeta_{n-1}, \cdots, \zeta_2, \zeta_1) \quad (21)$$

Now we can take  $Im \zeta_i \rightarrow 0$  on both sides and we get an equality between distributions for all  $\xi_i$ ,

$$W^{(n; \varphi)}(\xi_1, \xi_2, \cdots, \xi_{n-1}) = i^F (-1)^L W^{(n; i\varphi)}(\xi_{n-1}, \cdots, \xi_2, \xi_1). \quad (22)$$

Translated back into vacuum matrix elements this says

$$\begin{aligned} \langle 0 | \phi^{(p_1)}(x_1) \phi^{(p_2)}(x_2) \cdots \phi^{(p_n)}(x_n) | 0 \rangle = \\ i^F (-1)^L \langle 0 | \phi^{(p_n)}(-x_n) \cdots \phi^{(p_2)}(-x_2) \phi^{(p_1)}(-x_1) | 0 \rangle. \end{aligned} \quad (23)$$

Replacing  $(p_j)$  by  $(k_j, l_j)$  we have

$$\begin{aligned} \langle 0 | \phi^{(k_1, l_1)}(x_1) \phi^{(k_2, l_2)}(x_2) \cdots \phi^{(k_n, l_n)}(x_n) | 0 \rangle = \\ i^F (-1)^L \langle 0 | \phi^{(k_n, l_n)}(-x_n) \cdots \phi^{(k_2, l_2)}(-x_2) \phi^{(k_1, l_1)}(-x_1) | 0 \rangle. \end{aligned} \quad (24)$$

We can restore the original order of the fields on the right hand side by using the hermiticity of the scalar product,  $(\Psi, \Xi) = (\Xi, \Psi)^*$ . The appearance of complex

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<sup>7</sup>This condition is usually called the spin-statistics theorem. We have argued that in the present context it should be called the spin-locality theorem. [15]

conjugation is fine, since we know that  $\mathcal{CPT}$  is antiunitary. We find

$$\begin{aligned} \langle 0 | \phi^{(k_1, l_1)}(x_1) \phi^{(k_2, l_2)}(x_2) \cdots \phi^{(k_n, l_n)}(x_n) | 0 \rangle = \\ i^F (-1)^L \langle 0 | \phi^{(k_1, l_1)\dagger}(-x_1) \phi^{(k_2, l_2)\dagger}(-x_2) \cdots \phi^{(k_n, l_n)\dagger}(-x_n) | 0 \rangle^*, \end{aligned} \quad (25)$$

where  $F = \sum_1^n f_i$  and  $f$  is zero for a Bose field (with  $k + l$  even) and one for a Fermi field (with  $k + l$  odd). Now we can read off what  $\mathcal{CPT}$ , which for brevity we call  $\Theta$ , must be,

$$\Theta \phi^{(k, l)}(x) \Theta^\dagger = (-1)^l (\pm i)^f \phi^{(k, l)\dagger}(-x). \quad (26)$$

When we embed this relation in an arbitrary vacuum matrix element and use the invariance of the vacuum,  $\Theta | 0 \rangle = | 0 \rangle$ , we find precisely Eq.(25)! When we run this sequence of relations the other way, we conclude that weak local commutativity in the neighborhood of a Jost point is necessary and sufficient for  $\mathcal{CPT}$ . There is a sign ambiguity because  $F$ , the total number of Fermi fields in a nonvanishing vacuum matrix element, must be even.

Note that  $\mathcal{CPT}$  takes each irreducible of  $L_+^\dagger$  to a phase times itself, so that for example the part  $\phi^{(1,0)}$  of the Dirac spin field is mapped to  $i\phi^{(1,0)}$  and the part  $\phi^{(0,1)}$  is mapped to  $-i\phi^{(0,1)}$ . Both the vector and axial vector fields have the form  $\phi^{(1,1)}$  so these fields are indistinguishable under  $\Theta$  and both get the phase  $(-1)$  under  $\Theta$ . The analogous statements hold for the scalar and pseudoscalar fields,  $\phi^{(0,0)}$ , which both get phase 1. The antisymmetric rank two tensor field  $T^{\mu\nu}$  is  $\phi^{(2,0)} + \phi^{(0,2)}$  and also gets phase 1. The traceless symmetric tensor of rank two is  $\phi^{(2,2)}$  and also gets phase 1.

The  $\mathcal{CPT}$  operator  $\Theta$  interchanges undotted and dotted indices, so that

$$\phi^{(k, l)\dagger}(x) = \phi^{\dagger(l, k)}(x). \quad (27)$$

Under  $\Theta$  particles and antiparticles are interchanged (some particles may be identical to their antiparticles). Energies and momenta stay the same; spin components and helicities are reversed.

When we act twice by  $\Theta$  we find

$$\begin{aligned} \Theta^2 \phi^{(k, l)}(x) \Theta^{\dagger 2} &= \Theta (-1)^l (\pm i)^f \phi^{(k, l)\dagger}(-x) \Theta^\dagger \\ &= \Theta (-1)^l (\mp i)^f \Theta \phi^{(k, l)}(-x) \Theta^\dagger \\ &= (-1)^f \phi^{(k, l)}(x), \end{aligned} \quad (28)$$

so  $\Theta^2$  commutes with Bose fields and anticommutes with Fermi fields. The reader can check that the phase of  $\Theta^2$  cannot be changed by changing a phase in the definition of  $\Theta$ . This is true for all antiunitary operators.

## 6 $\mathcal{CPT}$ for the $S$ -matrix.

Because  $\Theta$  reverses time, in and out states are interchanged. Taking the antiunitarity of  $\Theta$  into account, the  $S$ -matrix obeys

$$S_{\alpha,\beta} \equiv_{out} \langle \alpha | \beta \rangle_{in} =_{out} \langle \hat{\beta} | \hat{\alpha} \rangle_{in} = S_{\hat{\beta},\hat{\alpha}}, \quad (29)$$

where  $|\hat{\alpha}\rangle$  has particles and antiparticles exchanged, spin components and helicities reversed, and energies and momenta the same as in  $|\alpha\rangle$

In terms of the  $S$ -operator this is

$$\Theta S \Theta^\dagger = S^{-1}, \text{ or } \Theta S = S^{-1} \Theta. \quad (30)$$

## 7 Summary.

Jost's general proof of the  $\mathcal{CPT}$  theorem leads directly to a definition of the  $\mathcal{CPT}$  symmetry applied to fields belonging to an arbitrary irreducible representation of  $SL(2, C)$ , the covering group of the real proper orthochronous Lorentz group,  $L_+^\uparrow$ . In the real Lorentz group spacetime inversion is not connected to the identity; however in the complex Lorentz group  $x \rightarrow -x$  is connected to the identity. Using the Wightman analytic functions that are analytic continuations of vacuum matrix elements of products of the fields, the Bargmann-Hall-Wightman theorem allows analytic continuation of the Wightman functions from (the covering group) of  $L_+^\uparrow$  to (the covering group of) the complex Lorentz group  $L_+(C)$  which now allows spacetime inversion. In the larger domain of analyticity given by the Bargmann-Hall-Wightman theorem there are real points of analyticity, the Jost points; however we cannot take limits to get a general relation between the the matrix elements at the original points and at the spacetime inverted points. However, if we invert the order of the fields, we can get the general relation. This is the only step where

we must assume something beyond Lorentz covariance: we must assume weak local commutativity at Jost points to allow the reordering. We can restore the original order of the fields by using the hermiticity of the scalar product, which is not an additional assumption. This step complex conjugates the matrix elements which means, as we expect, that  $\mathcal{CPT}$  is antiunitary rather than unitary. Now we are able to read off the general, simple result Eq. (26) for  $\mathcal{CPT}$  on each irreducible of (the covering group of) the Lorentz group,  $L_+^\uparrow$ .

## A Alternative description of the irreducible representations of the Lorentz group

An alternative way to describe the irreducible representations of the Lorentz group is to combine the rotation generators,  $J_i$ , with the boost (pure Lorentz transformation) generators,  $K_i$ , into a pair of commuting  $SU(2)$  generators  $A_i$  and  $B_i$ . The irreducible fields are  $\psi^{(A,B)}$  where  $A$  and  $B$  are the spins associated with each of the  $SU(2)$  algebras [13]. The relation between these two descriptions is  $\phi^{(2A,2B)} = \psi^{(A,B)}$ . The reader can check that the adjoint of  $\phi^{(k,l)}$  transforms as a field  $\chi^{(l,k)}$ ; that is the dotted and undotted indices get interchanged. For this reason we don't have to talk about adjoints of the fields; they are taken care of if we allow an arbitrary irreducible—they don't introduce anything new.

## B Detailed form of the transformation law for the fields

$$U(a, A) \phi_{\alpha_1 \dots \alpha_k \dot{\beta}_1 \dots \dot{\beta}_l}^{(k,l)}(x) U^\dagger(a, A) = A_{\alpha_1 \alpha'_1}^{-1} \dots A_{\alpha_k \alpha'_k}^{-1} A_{\dot{\beta}_1 \dot{\beta}'_1}^{-1*} \dots A_{\dot{\beta}_l \dot{\beta}'_l}^{-1*} \phi_{\alpha'_1 \dots \alpha'_k \dot{\beta}'_1 \dots \dot{\beta}'_l}^{(k,l)}(\Lambda(A)x + a), \quad (31)$$

for  $\Lambda(A) \in L_+^\uparrow$ , and

$$U(a, A, B)\phi_{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_l}^{(k,l)}(x)U^\dagger(a, A, B) = \\ A_{\alpha_1 \alpha'_1}^{-1} \dots A_{\alpha_k \alpha'_k}^{-1} B_{\beta_1 \beta'_1}^T \dots B_{\beta_l \beta'_l}^T \phi_{\alpha'_1 \dots \alpha'_k \beta'_1 \dots \beta'_l}^{(k,l)}(\Lambda(A, B)x + a), \quad (32)$$

for  $\Lambda(A, B) \in L_+(C)$ , where  $\phi$  is symmetric in the  $\alpha$ 's and in the  $\beta$ 's separately for both cases.

## C Qualitative difference nature of domains of analyticity for functions of several complex variables

Readers can ignore this appendix which is not necessary for our discussion of the  $\mathcal{CPT}$  theorem. Functions of several complex variables differ qualitatively from functions of a single complex variable in their possible domains of analyticity. For a single complex variable, for every domain in the complex plane bounded by a smooth curve there is an analytic function that cannot be continued outside this domain. Thus any such region is the domain of analyticity for some function. For several complex variables this is not true. Domains of analyticity must be “holomorphically convex.” Intuitively such domains must not have “dimples” that can be removed by analytically continuing *any* analytic function across the dimple using a Cauchy contour that surrounds the dimple. The additional dimensions available for several complex variables is what allows this analytic continuation. This possibility of analytic continuation comes into play when the commutativity or anticommutativity of fields at spacelike separation is imposed on the Wightman functions. In the case of interest for the  $\mathcal{CPT}$  theorem, which involves reversing all the fields in the vacuum matrix element, the new and old extended tubes agree, so further analytic continuation is not possible.

## D Lorentz covariance alone does not suffice

Lorentz covariance alone is not sufficient for  $\mathcal{CPT}$ . One example suffices to show this. A free or generalized free field can be Lorentz covariant but not obey  $\mathcal{CPT}$  invariance if the particle and antiparticle masses are different [16]. What fails in that case is that WLC does not hold at Jost points. This possibility is associated with the purely timelike support in momentum space of free or generalized free fields; for timelike momenta positive and negative energies can be separated in a covariant way. By contrast positive and negative energies can be transformed into each other for spacelike momenta. Note that although the fields in these examples transform covariantly their time-ordered products are not covariant. Thus if we require that time-ordered products be covariant as part of Lorentz covariance of a theory then, as shown in [16], free fields that violate  $\mathcal{CPT}$  are not covariant. See [17] for a detailed analysis of hybrid Dirac fields (“homeotic” fields [18]) which can be covariant only when they are non-interacting but even in the free case have time-ordered products that are not covariant

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